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Asymptotic fields of stress and damage of a mode I creep crack in steady-state growth

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Abstract

Asymptotic fields of stress, strain rate and damage of a mode I creep crack in steady-state growth are analyzed on the basis of Continuum Damage Mechanics by employing a semi-inverse method. A damage field $D(x)$ for steady-state crack growth represented by an undetermined power function r^l of radius r from the crack tip is assumed first, and the corresponding asymptotic stress field of a mode I crack in a non-linear creep damage material is analyzed by solving a two-point boundary value problem of non-linear differential equations. Then, the exponent l of undetermined damage field is determined so that the assumed damage field $D(x)$ may be consistent with the resulting asymptotic stress field and the damage evolution equation. Finally, the relations between exponent p of the asymptotic stress distribution and exponents n and m of power creep constitutive law and the power creep damage law are elucidated. The effects of material damage on the crack-tip stress field in non-linear materials are discussed in detail. Comparison between the results of the present analysis and those of earlier papers of fracture mechanics on the creep-crack growth analyses based on grain-boundary cavitation is also made.

Keywords: Damage mechanics; Creep damage; Mode I crack; Crack-tip; Stress field; Damage field; Stress singularity; Fracture; HRR stress field

1. Introduction

Though the HRR field (Hutchinson, 1968; Rice and Rosengren, 1968) has been obtained for an ideal discrete crack in intact non-linear hardening materials, the fracture process in usual ductile materials is

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brought about by nucleation, growth and coalescence of distributed microscopic cavities in front of the crack-tip, and this damage field gives significant influence on stress field near the crack-tip. Therefore, analyses of effects of material damage on the stress and strain fields near crack-tip in non-linear materials provide very important problems not only for evaluation of crack behaviour in materials but also for discussion of stability and convergency of numerical analyses. In this context, these problems have been discussed in a number of papers, especially for elastic–plastic-brittle cracks (Bui and Ehrlacher, 1980), elastic–plastic cracks (Knowles and Sternberg, 1980, 1981; Wang and Chow, 1992; Zhang et al., 1993; Gao and Bui, 1995; Benallal and Siad, 1997), creep cracks (Bassani and Hawk, 1990; Astafjev et al., 1991; Astafjev and Grigorova, 1995; Lu et al., 1997) and fatigue cracks (Zhao and Zhang, 1994).

In the case of creep cracks, in particular, the asymptotic stress fields at the crack-tips in materials subject to damage were analyzed by Astafjev et al. (1991) and Astafjev and Grigorova (1995) for mode I and mode III creep cracks in steady-state growth, and by Lu et al. (1997) for a mode I stationary crack. However, because of difficulties of stress-damage coupled analysis, they could not derive complete and consistent results. Namely, Astafjev and Grigorova (1995) could not satisfy the boundary conditions of a sharp crack in their numerical analysis, and obtained an approximate solution by replacing the boundary conditions by those of a V-notch; this replacement may lead to a significant error in stress singularity at the crack-tip (Williams, 1952). Though Lu et al. (1997), on the other hand, made a priori assumption of vanishing stress components at the crack tip, this assumption may give too much constraints to the analysis; i.e., these analyses may need further refinement.

One of the major causes in difficulty in the analyses of crack tip fields in damaged or strain softening materials is concurrence of the elliptic and hyperbolic regimes of the governing field equations together with related discontinuity in deformation gradient and stress (Knowles and Sternberg, 1980, 1981; Benallal and Siad, 1997). Moreover, the mathematical structure of governing equations may be affected largely by the modeling of damage.

Bassani and Hawk (1990), furthermore, analysed the influence of creep damage on crack-tip fields under small-scale-creep conditions by the use of finite element method. However, because their analysis is numerical and is concerned with a blunt crack, systematic information on the effect of material damage on the asymptotic crack-tip field is not available from the analysis.

Because of the engineering importance, a number of papers discussed the problems of creep-crack growth also from the view point of fracture mechanics and micromechanics (Riedel and Rice, 1980; Hui and Riedel, 1981; Riedel, 1987; Hutchinson, 1983). However, most of these analyses are not only one-dimensional and are focused on a plane in front of the crack-tips, but also disregard the effect of damage on the stress fields. Therefore, systematic and three-dimensional analyses of the crack-tip field in materials undergoing creep damage are hardly available in these papers of fracture mechanics.

The present paper is concerned with an elaborate continuum damage mechanics analysis of the asymptotic fields near a mode I creep crack in steady-state growth; damage field and its effects on the asymptotic stress- and strain-rate fields at the crack are solved by the semi-inverse method. Namely, a damage field for steady-state crack growth represented by an undetermined power function r^l of radius r from the crack tip is assumed first, and the corresponding asymptotic fields of stress and strain-rate of a mode I creep crack are analyzed by employing power-law creep damage theory (Kachanov, 1986; Lemaitre and Chaboche, 1990; Lemaitre, 1996). Then, the exponent l of the undetermined damage field is determined so that the assumed damage field may be consistent with the resulting stress field and the damage evolution equation. Finally, the relations between exponent p of the asymptotic stress field $\sigma_{ij} = K\tilde{\sigma}_{ij}(\theta)r^p$ at the crack tip and exponents n and m of power creep constitutive law and the power creep damage equation are elucidated. The effects of material damage on the crack-tip stress field for non-linear materials are discussed in some detail. Comparison between the results of present analysis and those of earlier papers of fracture mechanics on creep-crack growth analyses based on grain-boundary cavitation is also discussed.

2. Governing equations and asymptotic stress field

2.1. Governing equations

Let us take a mode I creep crack extending at a constant rate v in a stationary cartesian coordinate system $O - X_1X_2X_3$ as shown in Fig. 1, and assume that the material in the vicinity of the crack is in the state of plane strain or of plane stress. Then, we further move cartesian coordinates $o - x_1x_2x_3$ and polar coordinates $o - r\theta z$ with origins o at the tip of moving cracks, where the direction x_1 and that of $\theta = 0$ are in the direction of crack extension. By denoting the components of stress and strain with respect to moving coordinates by σ_{ij} and ε_{ij}, the governing equations for a mode I creep crack in steady-state extension are given as follows:

Components of stress

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} \tag{1a}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} \tag{1b}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) \tag{1c}$$

where $\Phi = \Phi(r, \theta, z)$ is the Airy stress function.

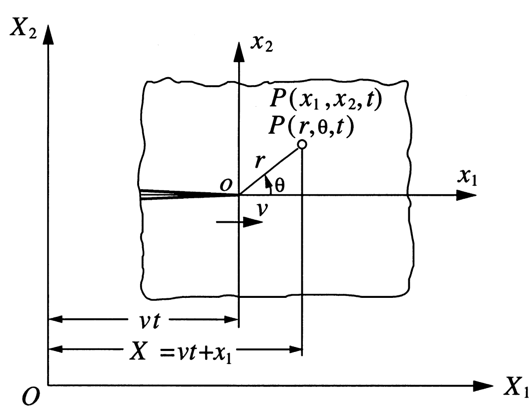


Fig. 1. Stationary and moving coordinate systems.

Condition of compatibility

$$r\frac{\partial}{\partial r}\left(\frac{\partial(r\dot{\varepsilon}_{\theta\theta})}{\partial r}\right)+\frac{\partial^2\dot{\varepsilon}_{rr}}{\partial\theta^2}-r\frac{\partial\dot{\varepsilon}_{rr}}{\partial r}-2\frac{\partial}{\partial r}\left(r\frac{\partial\dot{\varepsilon}_{r\theta}}{\partial\theta}\right)=0 \tag{2a}$$

$$\dot{(\cdot)}=\frac{\partial}{\partial t}-\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)v \tag{2b}$$

where $\dot{(\cdot)}$ denotes the material time derivative with respect to time t.

Constitutive equation of creep

If we represent the damage state of material by an isotropic damage variable $D(0\leq D\leq 1)$, and employ the hypothesis of strain equivalence in damage mechanics, the creep constitutive equation of a damaged material can be given as follows (Kachanov, 1986; Lemaitre and Chaboche, 1990; Lemaitre, 1996):

$$\dot{\varepsilon}_{ij}=\frac{3}{2}\frac{A\sigma_{\mathrm{EQ}}^{n-1}s_{ij}}{(1-D)^n}\quad(i,j=1,2,3) \tag{3}$$

where $s_{ij}=\sigma_{ij}-(1/3)\sigma_{kk}\delta_{ij}$ and $\sigma_{\mathrm{EQ}}=(3s_{ij}s_{ij}/2)^{1/2}$ are the deviatoric stress and the equivalent stress, respectively. The symbols n and A are the creep exponent and material constant, respectively.

Evolution equation of creep damage

By assuming that the creep damage is governed by the equivalent stress, the damage evolution equation in multi-axial state of stress may be given as follows (Kachanov, 1986; Lemaitre and Chaboche, 1990; Lemaitre, 1996):

$$\dot{D}=B\left(\frac{\sigma_{\mathrm{EQ}}}{1-D}\right)^m \tag{4}$$

where $D(r, \theta, z, t)$ is the damage variable, while B and m $(m>0)$ denote material constants. In the particular case of steady-state crack growth, we have

$$\frac{\partial D}{\partial t}\equiv 0 \tag{5}$$

and hence Eq. (4) leads to

$$-\cos\theta\frac{\partial D}{\partial r}+\frac{\sin\theta}{r}\frac{\partial D}{\partial\theta}=\frac{B}{v}\left(\frac{\sigma_{\mathrm{EQ}}}{1-D}\right)^m \tag{6}$$

2.2. Asymptotic stress field

Since the present paper aims to elucidate the effect of material damage on the asymptotic fields near the creep-crack tip, we will assume the following asymptotic solution for the crack-tip stress:

$$\Phi(r,\theta)=Kr^s f(\theta) \tag{7}$$

where K and s are undetermined constants, while $f(\theta)$ is an unknown function of θ. By substituting Eq. (7) into Eq. (1), we have the components of the asymptotic stress field as follows:

$$\sigma_{rr}(r,\theta) = Kr^{p}\left[s\cdot f(\theta) + f''(\theta)\right] = Kr^{p}\tilde{\sigma}_{rr}(\theta)$$

$$\sigma_{\theta\theta}(r,\theta) = Kr^{p}\left[s(s-1)f(\theta)\right] = Kr^{p}\tilde{\sigma}_{\theta\theta}(\theta)$$

$$\sigma_{r\theta}(r,\theta) = Kr^{p}\left[(1-s)f'(\theta)\right] = Kr^{p}\tilde{\sigma}_{r\theta}(\theta)$$

$$\sigma_{\mathrm{EQ}}(r,\theta) = Kr^{p}\tilde{\sigma}_{\mathrm{EQ}}(\theta) \tag{8}$$

where $p = s-2$ represents the exponent of stress field, and $\tilde{\sigma}_{rr}(\theta),\ldots,\tilde{\sigma}_{\mathrm{EQ}}(\theta)$ are given by expressions in the corresponding brackets [ ] in Eq. (8). In view of Eq. (8), the undetermined constant K corresponds to the stress intensity factor of a non-linear material and depends on exponent n of the creep constitutive equation (3).

3. Semi-inverse solution of non-linear differential equations

In order to determine the asymptotic stress field at the crack-tip, we will have to determine first the damage field $D(r,\theta)$ by substituting the stress of Eq. (8) into Eq. (6). Once $D(r,\theta)$ were determined, then by substituting the creep rate $\dot{\varepsilon}_{ij}$ of Eq. (3) obtained from the resulting D and stresses of Eq. (8) into the equation of compatibility (2), a set of non-linear ordinary differential equations for the unknown function $f(\theta)$ could be derived. However, it is difficult in general to derive a rigorous solution of $D(r,\theta)$ from Eq. (6) throughout the whole region of analysis.

Thus, in the present analysis, we will determine by semi-inverse method an asymptotic damage field which satisfies the evolution equation (6) in the region in front of the crack-tip. Namely, by referring to experimentally observed damage field (Liu et al., 2000), we first postulate an elliptic damage distribution represented by a power function $h(\theta)r^{l}$ of radius r as shown in Fig. 2. Then, the asymptotic stress field can be obtained by the use of resulting damage field and Eqs. (2), (3) and (8), and finally the exponent l

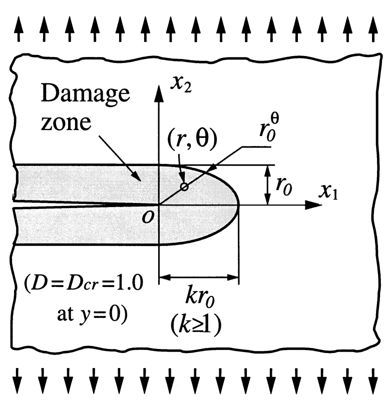


Fig. 2. Mode I crack and damage field.

of the undetermined damage field can be determined so that the postulated damage field together with the relevant stress field may satisfy the damage evolution equation (6).

3.1. Elliptic damage distribution at crack-tip

According to the experimental observation on the damage distribution around a mode I creep crack in OFHC copper at 250°C in steady-state growth (Liu et al., 2000), contour of the damage field can be represented by a semi-ellipse in front of the crack and by a wake parallel to the crack plane behind the crack.

In view of this observation, we will now assume the damage distribution of Fig. 2 represented as follows:

$$1 - D = h(\theta)\left(\frac{r}{r_0}\right)^l \quad (0 \le l < 1) \tag{9a}$$

$$h(\theta) = \begin{cases} \left[(\cos\theta/k)^2 + (\sin\theta)^2\right]^{l/2} & 0 \le \theta \le \pi/2 \\ (\sin\theta)^l & \pi/2 < \theta \le \pi \end{cases} \tag{9b}$$

where l and r_0 are parameters characterizing the damage distribution, while $h(\theta)$ and k denote the θ-distribution of the damage field and its aspect ratio. The locus $r = r_0 h(\theta)^{-1/l}$, in particular, represents the boundary of the damage field where $D = 0$, and hence is the boundary between the damaged and the undamaged region.

3.2. Differential equations of the asymptotic stress field for elliptic damage distribution

As described already, by substituting the stress and damage fields of Eqs. (8) and (9) into the creep constitutive equation (3), and then by substituting the resulting creep rate $\dot{\varepsilon}_{ij}$ into the equation of compatibility (2), we can readily obtain the differential equations governing unknown function $f(\theta)$. The differential equations can be written for the states of plane strain and plane stress as follows:

State of plane strain

$$\left[\frac{\partial}{\partial\theta^2} - n(s-l-2)\{n(s-l-2)+2\}\right]\left[h(\theta)^{-n}\tilde{\sigma}_{EQ}^{n-1}\{s(2-s)f'(\theta) + f''(\theta)\}\right] + 4(s-1)[n(s-l-2)+1]\frac{\partial}{\partial\theta}\left[h(\theta)^{-n}\tilde{\sigma}_{EQ}^{n-1}f'(\theta)\right] = 0 \tag{10a}$$

where

$$\tilde{\sigma}_{EQ} = \left\{3[s(2-s)f(\theta) + f''(\theta)]^2/4 + 3(1-s)^2 f'(\theta)^2\right\}^{1/2} \tag{10b}$$

State of plane stress

$$\left[n(s-l-2)-\frac{\partial}{\partial\theta^2}\right]\left[h(\theta)^{-n}\tilde{\sigma}_{\mathrm{EQ}}^{n-1}\{s(s-3)f(\theta)-2f''(\theta)\}\right]+\left[n(s-l-2)+1\right]n(s-l-2)h(\theta)^{-n}\tilde{\sigma}_{\mathrm{EQ}}^{n-1}\left[s(2s-3)f(\theta)-f''(\theta)\right]+6\left[n(s-l-2)+1\right](s-1)\frac{\partial}{\partial\theta}\left[h(\theta)^{-n}\tilde{\sigma}_{\mathrm{EQ}}^{n-1}f'(\theta)\right]=0 \tag{11a}$$

where e

$$\tilde{\sigma}_{\mathrm{EQ}}(\theta)=\left[s^2\left(3-3s+s^2\right)f(\theta)^2+\left(3-6s+3s^2\right)f'(\theta)^2+s(3-s)f(\theta)f''(\theta)+f''(\theta)^2\right]^{1/2} \tag{11b}$$

3.3. Boundary conditions of differential equations and their numerical solutions

Because of the symmetry of the stress components σ_{rr} and $\sigma_{\theta\theta}$ at $\theta=0$ and of the vanishing condition of the stress components $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ at the crack plane $\theta=\pi$, we have the following two boundary conditions for the differential equations (10) and (11):

$$f'(0)=0, \qquad f'''(0)=0 \tag{12}$$

$$f(\pi)=0, \qquad f'(\pi)=0 \tag{13}$$

The two-point boundary problem of Eqs (10)–(13) for the asymptotic stress field $f(\theta)$ can be solved by a Shooting Method (Hutchinson, 1968) as an initial value problem. For this purpose, the initial values of $f(0)$ and $f''(0)$ should be specified besides the initial conditions of Eq. (12). Since the differential equations (10) and (11) are homogeneous with respect to function $f(\theta)$, the value of $f(0)$ can be specified arbitrarily. As regards $f''(0)$, on the other hand, we will specify a proper undetermined constant β. Then, besides the condition (12), additional initial conditions

$$f(0)=1, \qquad f''(0)=\beta \tag{14}$$

are prescribed for the differential equations (10) and (11).

In order to calculate the differential equations (10) and (11) by a shooting method for a given damage field (i.e., for given l and $h(\theta)$), Eqs. (10) and (11) can be readily integrated numerically if the values of the exponent s in Eq. (8) and the undetermined constant β of Eq. (14) are given properly. Thus, the values of s and β can be determined as the eigen values s^* and β^* of the differential equations such that the numerical solution $f(\theta)$ may satisfy the boundary conditions (13) on the crack plane:

$$f(\pi)=F_1(s,\beta)=0$$

$$f'(\pi)=F_2(s,\beta)=0 \tag{15}$$

Solution $f(\theta)$ can be determined as an eigen function of the eigen values s^* and β^*.

The numerical integration of differential equations (10) or (11) to obtain $f(\theta)$ are performed by the fourth-order Runge–Kutta method, while the solution of non-linear simultaneous equations for s^*, β^*

are obtained by the Gauss–Newton least-square method so that they satisfy the condition $f(\pi)^2 + f'(\pi)^2 < 10^{-6}$.

The eigen values s^* and β^* which satisfy the condition (15) are not unique in general. Thus, we will obtain only the solution which corresponds to the lowest exponent s, and this lowest exponent s is subjected to the following condition of the bounded stress work done in a material element including crack tip:

$$s > \frac{(2+l)n}{(n+1)} \tag{16}$$

3.4. Coupled solution of the asymptotic stress and damage fields

In the above analysis, the two-point boundary value problem of differential equations (10) or (11) together with the boundary conditions (12)–(15) was calculated for the prescribed damage field of Eq. (9), and gave the relation among the exponent of the damage distribution l, the creep exponent n and exponent s (or p) of the stress distribution. However, the damage field (9) is not consistent with the damage evolution equation (6) and the resulting asymptotic stress field.

Since the determination of the damage field which satisfies the damage evolution equation for the entire region of the problem is difficult, we will satisfy this condition approximately by specifying the damage field of Eq. (9) so that it may be consistent with the evolution equation (6) only in the region in front of the crack because this region gives largest influence on the asymptotic fields.

On the crack plane $\theta = 0$ in front of the crack, the evolution equation (6) is reduced to

$$\frac{B}{v}\left(\frac{\sigma_{\mathrm{EQ}}}{1-D}\right)^m = -\frac{\partial D}{\partial r} \tag{17}$$

Substitution of Eqs. (8) and (9a) into this equation gives

$$\frac{B}{v}\frac{1}{l}(kr_0)^{l(m+1)}\left[K\tilde{\sigma}_{\mathrm{EQ}}(0)\right]^m = r^{(m+1)l-pm-1} \tag{18}$$

In order that this condition could be satisfied always, we have the following relations:

$$l = \frac{pm+1}{m+1} \tag{19a}$$

$$v = \frac{B}{l}(kr_0)^{l(m+1)}\left[K\tilde{\sigma}_{\mathrm{EQ}}(0)\right]^m \tag{19b}$$

which specifies the exponent l of the damage distribution (9) and the steady-state crack rate v. Eq. (19b) implies that the crack rate v is proportional to the coefficient B of the damage evolution equation (6) and to the stress $[K\tilde{\sigma}_{\mathrm{EQ}}(0)]^m$. As observed from Eq. (9a), exponent of damage distribution has the value $0 \leq l < 1$, and hence Eq. (19b) predicts the infinite crack rate $v = \infty$ in the case of uniform damage $l = 0$.

The stress and the damage field in front of an extending crack together with the ensuing creep-crack growth rate depends significantly on the level of applied load (Riedel and Rice, 1980; Riedel, 1987). The creep-crack growth rate v under HRR field has been reported to be characterized by the non-linear fracture mechanics parameter C^* (Riedel, 1987; Chung et al., 1990). It should be noted that Eq. (19b) implies that creep-crack rate v is related to the exponent m of the creep damage evolution equation (4) in addition to the creep exponent n of Eq. (3), and shows clear contrast with the results of earlier papers because C^* is characterized by the creep exponent n.

4. Results of analysis and discussions

4.1. Effects of material damage on the stress singularity at crack-tip

In the numerical analysis of Section 3.3, the eigen value s (and hence the exponent of the stress field $p = s - 2$) has been obtained for given sets of the damage exponent l and the creep exponent n as parameters; i.e., we have $p - l - n$ relation among the exponents of p, l and n. The detailed numerical procedure to obtain $p - l - n$ relation and its explicit result have been reported in the previous paper of the authors (Liu and Murakami, 1998).

Eq. (19a) of Section 3.4, on the other hand, specifies the consistent values of l as a function of the exponent p of stress distribution and exponent m of the damage evolution equation; i.e. $p - l - m$ relation. By the use of these two relations of $p - l - n$ and $p - l - m$, we can readily calculate the value of exponent p as a function of the creep exponent n and damage exponent m.

The small circles in Fig. 3 show the numerical results for the exponent p of the asymptotic stress field (8) of mode I creep crack in steady-state growth, for the case of $k = 1$ in Fig. 2. The dotted lines, on the other hand, represent the exponent p of the HRR stress field for undamaged non-linear materials:

$$p = -\frac{1}{n+1} \tag{20}$$

Since the exponent $p < 0$ implies the stress singularity at the crack-tip $r = 0$, the HRR field has always stress singularity for any finite value of the stress exponent n. However, as observed in Fig. 3, the presence of material damage increases significantly the value of the exponent p, and may give non-singular stress field even for finite values of n. Namely, Fig. 3 shows that the stress singularity of the asymptotic field is determined by the relative values of n and m, and this is a very important result as the effect of the material damage on the asymptotic field at a crack-tip.

By comparing Fig. 3(a) and (b), it will be observed that, for a given set of n and m, the case of plane strain has larger stress singularity than that of the plane stress, which is different from the cases of undamaged non-linear materials (Hutchinson, 1968; Rice and Rosengren, 1968).

In the fracture tests of ductile materials, the fracture toughness of cracked thick specimens is known to be lower than that of thin specimens. This is usually attributed to the triaxial stress at the crack tip induced in the state of plane strain. Fig. 3 shows that, when a damage field exists at the crack tip, besides the triaxial stress, plane strain state has larger stress singularity. Thus, it should be noted that, in view of material damage, the fracture toughness of ductile materials in the state of plane strain may be further decreased from that of plane stress state.

Finally, the solid lines in Fig. 3 represent the approximate expression to the corresponding numerical results, and are given by the following relations:

$$p = \frac{n^{(1+c/n)} - (m+1)}{n + m[1 + n - n^{(1+c/n)}] + 1} \tag{21a}$$

$$c = 0.100 \quad \text{(plane strain state)}$$

$$c = 0.370 \quad \text{(plane stress state)} \tag{21b}$$

Eq. (21) coincides with the numerical results within the error of 0.01 (plane strain state) and 0.02 (plane stress state) except for a few results of $m = 1$.

4.2. Criterion of the stress singularity at crack tip in the state of plane strain

According to the previous paper of the authors (Liu and Murakami, 1998) (where symbol m was used instead of l) in the state of plane strain, the $p - l - n$ relation described in Section 4.1 can be expressed by the following approximate relation:

$$p = \frac{ln - 1}{n + 1} \tag{22}$$

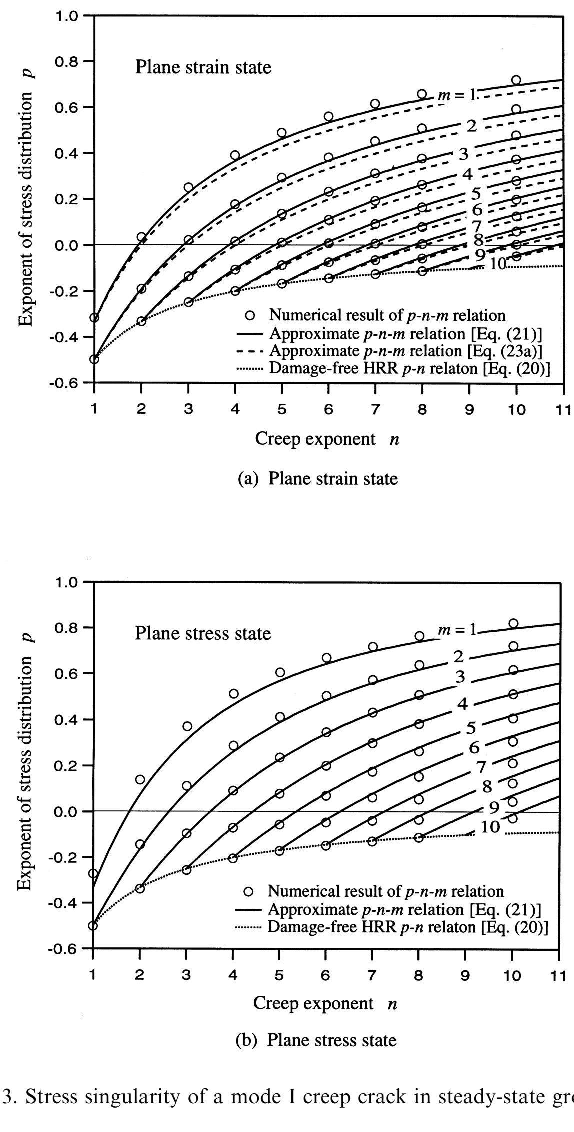


Fig. 3. Stress singularity of a mode I creep crack in steady-state growth.

By substituting Eq. (22) into Eq. (19a) we have l , and thus substitution of this l into Eq. (22) gives the following relations:

$$p = \frac{n-(m+1)}{n+m+1} \tag{23a}$$

$$l = \frac{n-(m-1)}{n+m+1} \tag{23b}$$

Relations (23a) and (23b) give the approximate relations among the creep exponent n, the damage exponent m, the exponent of stress distribution p and exponent l of the damage distribution consistent to the stress distribution of Eq. (8).

Since the damage variable has the range $0 \leq D \leq 1$, Eq. (9) specifies the exponent l to be $l \geq 0$. Then, Eq. (23b) imposes the condition

$$n \geq m-1 \tag{24}$$

which is usually satisfied because the relation

$$n \geq m \tag{25}$$

is ascertained for most metallic materials (Kachanov, 1986).

According to Eqs. (23a) and (24), the value of the exponent of stress distribution p is

$$p < 0, \qquad m-1 \leq n < m+1 \tag{26a}$$

$$p \geq 0, \qquad m+1 \leq n \tag{26b}$$

Namely, this result implies that the asymptotic stress field at a crack tip may be singular or non-singular when the set of exponents n and m is in the range of Eq. (26a) or Eq. (26b). It should be noted once more that the condition (26a) is subject to the limitation of a physical requirement (25) for engineering metallic materials.

The result of Eq. (26) furnishes very important information not only for the evaluation and understanding of creep crack behavior in structural component, but also for the discussion of the stability and the convergency of its numerical analyses. Namely, the problem of mesh-dependence of numerical results is usually one of the most crucial problems in the local approach of fracture based on the finite element method and damage mechanics; besides the hyperbolicity of the governing equations, the singularity at the crack tip may be one of the major causes of the mesh dependence. Then, Eq. (26) may give a criterion for this purpose.

The result of Eq. (23a) is entered also in Fig. 3(a) by dashed lines. As observed in the figure, as regards the value of n corresponding to the bound of singularity $p = 0$, Eq. (23a) coincides with the corresponding numerical results within the error of 5%.

In the case of the plane stress, on the other hand, it is difficult to do a similar argument as above, because we could not obtain a simple expression for p as Eq. (22).

4.3. Effects of damage field on the state of asymptotic stress

Fig. 4(a) and (b) show the effects of a damage field on the radial distribution of the asymptotic stress field at the creep-crack tip in the case of a typical creep exponent $n = 5$. The solid and dashed lines were calculated by the stress field of Eq. (8) together with different values of the exponent p shown in Fig. 3. The dotted lines, on the other hand, represent the asymptotic stresses of the undamaged HRR fields. Though the lines of $m = 6$ coincides with that of the HRR field because the case of $m = n + 1$ corresponds to the undamaged materials, they have been eliminated in Fig. 4 because of the physical requirement of Eq. (25).

In view of Eq. (8), all the stress components σ_{ij} as well as the equivalent stress σ_{EQ} have an identical radial distribution, and these radial distributions are also same in all θ-directions.

Since the radius $r = r_0$ represents the boundary between the damaged and undamaged zone and implies $D = 0$, all the lines in Fig. 4 attain to an identical value of $\sigma_{ij}/(Kr_0^p\tilde{\sigma}_{ij}) = 1$ corresponding to the undamaged material (i.e., the value corresponding to HRR field) at $r/r_0 = 1$. When r/r_0 tends to 0, on the other hand, the stresses σ_{ij} for $m = 5$ tend to ∞ since the stress exponent p is negative in this case.

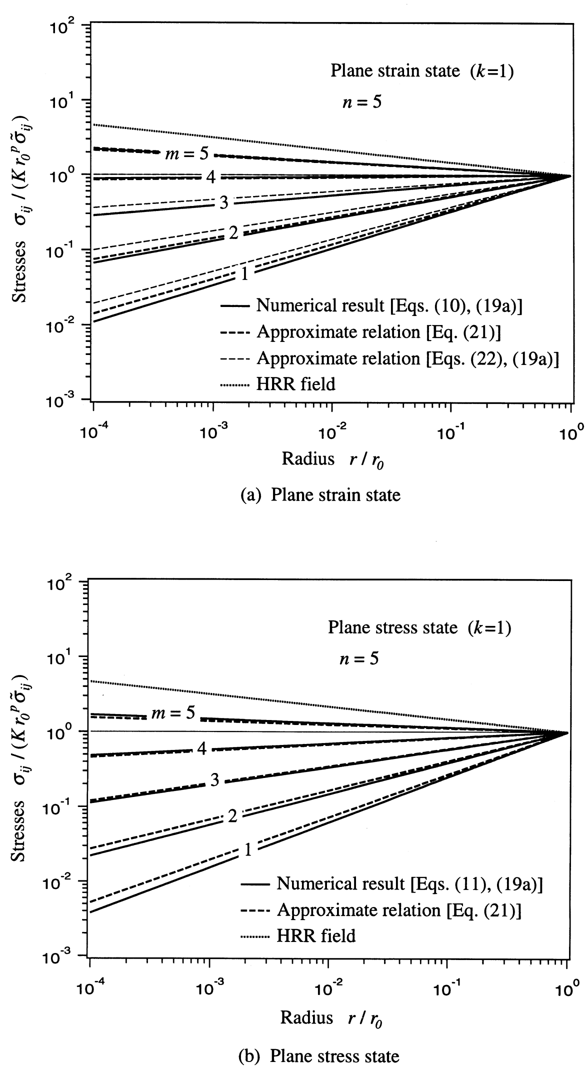
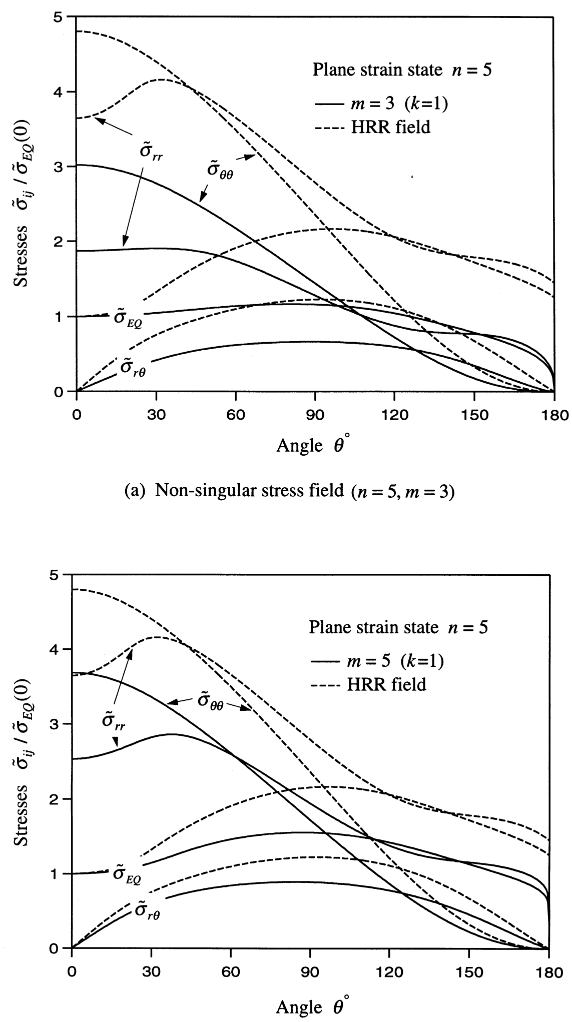


Fig. 4. Radial distribution of crack-tip stress.

Hitherto a number of models have been reported to analyze the process of creep-crack growth from the view point of damage mechanics (Bassani and Hawk, 1990; Astafjev et al., 1991; Astafjev and Grigorova, 1995; Lu et al., 1997) and of fracture mechanics and micromechanics (Riedel and Rice, 1980; Hui and Riedel, 1981; Hutchinson, 1983; Wu et al., 1986; Riedel, 1987; Chung et al., 1990). Besides that most of these analyses are one-dimensional, the effects of damage on the stress field have been hardly elucidated systematically. However, Bassani and Hawk (1990), among them, analysed damage and stress field at a mode I blunted creep-crack in plane strain state by finite element method by employing Kachanov-type creep damage constitutive equations. They demonstrated variation of the stress distribution from the elastic crack-tip field at the instant of loading through the stage of creep-crack growth. Their analysis show that after the crack starts to grow, the crack tip stress decreases towards



(a) Non-singular stress field ($n = 5, m = 3$)

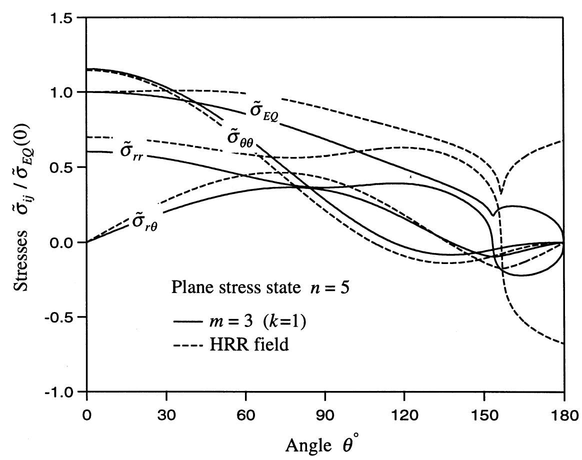
(b) Singular stress field ($n = 5, m = 5$)

Fig. 5. Circumferential distribution of crack-tip stress (plane strain state, $k = 1$).

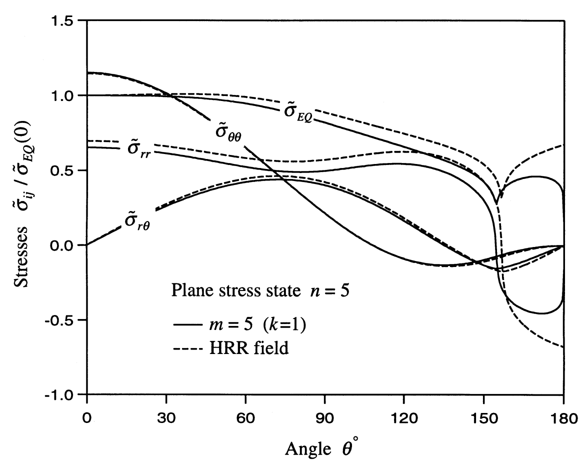
zero due to damage. Since the creep and the damage exponent in their analysis are $n = 5$ and $m = 3$, their results conform to the results of Fig. 4(a).

Figs. 5 and 6, on the other hand, show the θ-variation of the three stress components $\tilde{\sigma}_{rr}$, $\tilde{\sigma}_{\theta\theta}$ and $\tilde{\sigma}_{r\theta}$ and of the equivalent stress $\tilde{\sigma}_{\mathrm{EQ}}$ for the states of plane strain and plane stress, respectively. The HRR field has been entered by dotted lines for the sake of comparison. The cases of (a) and (b) in these figures correspond to the non-singular stress fields $(n = 5, m = 3)$ and the singular stress fields $(n = 5, m = 5)$, respectively. Because the magnitude of the coefficient K of Eq. (8) is unknown, the results of Figs. 5 and 6 have been normalized by the use of the equivalent stress $\tilde{\sigma}_{\mathrm{EQ}}(0)$.

In the case of the plane strain state of Fig. 5, though the stress components show smooth distribution, every component vanishes at $\theta = \pi$ because of damage. This is in contrast to HRR field shown by



(a) Non-singular stress field $(n = 5, m = 3)$

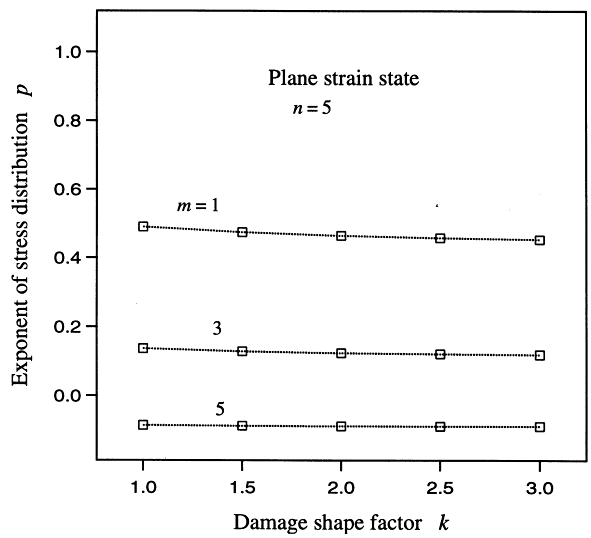


(b) Singular stress field $(n = 5, m = 5)$

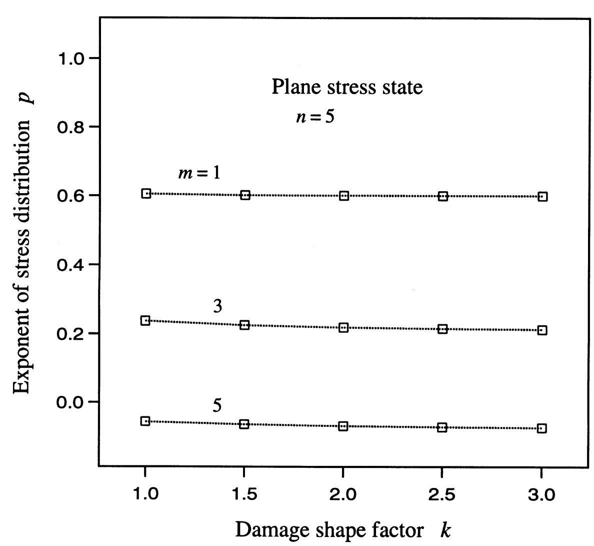
Fig. 6. Circumferential distribution of crack-tip stress (plane stress state, $k = 1$).

dotted lines. The stress components $\tilde{\sigma}_{\theta\theta}$ and $\tilde{\sigma}_{rr}$ in Fig. 5(b) have larger values at $\theta = 0$ than those of Fig. 5(a), and this may be accounted for by the singular stress field in Fig. 5(b).

As regards the plane stress state of Fig. 6(a) and (b), on the other hand, dominant singularity is observed in θ-distribution, and it is more salient in the case of singular stress field of Fig. 6(b). In both cases, all the stress components at $\theta = \pi$ vanish because of its complete damage.



(a) Plane strain state



(b) Plane stress state

Fig. 7. Effects of the shape of damage distribution.

4.4. Singularity of the strain rate field

By substituting the asymptotic stress of Eq. (8) and the damage distribution (9) into the creep constitutive equation (3), we have the expression of the creep rate

$$\dot{\varepsilon}_{ij} = \left(\frac{3}{2}\right) A K^n r^{(p-l)n} \tilde{\dot{\varepsilon}}_{ij}(\theta) \tag{27a}$$

where

$$\tilde{\dot{\varepsilon}}_{ij}(\theta) = \tilde{\sigma}_{\mathrm{EQ}}(\theta)^{n-1} \tilde{s}_{ij}(\theta) \left[\frac{r_0^l}{h(\theta)}\right]^n \tag{27b}$$

In view of $0 \le l < 1$, Eq. (22) gives

$$(p-l)n = -\frac{(l+1)n}{n+1} < 0 \tag{28}$$

and thus strain rates $\dot{\varepsilon}_{ij}$ is always infinite at the crack tip $r = 0$ irrespective of the nature of the asymptotic stress field. This may correspond exactly to the steady-state crack growth.

4.5. Effects of the shape of damage distribution

By referring to the experimental observations (Liu et al., 2000), the semi-inverse analysis of Chapter 3 was performed by postulating the damage distribution of Fig. 2. However, the numerical analysis of Section 4 has been performed for the special case of the shape factor $k = 1$; i.e., semi-circular damage field in front of the crack combined with a wake parallel to the crack plane.

In order to discuss the effect of shape of the damage region in some detail, we will now perform the calculation for five cases of shape factor $k = 1.0$–3.0. Fig. 7(a) and (b) shows the relations between the damage shape factor k and exponent of stress distribution p for a stress exponent $n = 5$, for the states of plane strain and plane stress. The results of Fig. 7 shows that the effects of damage shape factor k is insignificant for each case of the damage exponent m. This implies that the factor which governs singularity of the asymptotic stress field is local character of the damage field rather than the global geometry of the damage distribution.

5. Conclusions

The effects of material damage on the asymptotic stress field of a mode I creep crack in steady-state growth were analysed on the basis of continuum damage mechanics by postulating power law creep damage theory. The resulting governing differential equations were solved by semi-inverse method. The relations between exponent p of the asymptotic stress field and exponents n and m of the power law creep constitutive law and the power creep damage law were elucidated in detail. The results of the present analysis are summarized as follows:

1. While the HRR-field of non-linear fracture mechanics always shows the stress singularity at the crack-tip for any finite value of the creep exponent n, the preceding material damage in front of the crack tip decreases the singularity, and may give non-singular stress field.
2. In the particular case of mode I creep crack in the state of plane strain, by representing the asymptotic stress field at the crack-tip and the preceding damage field by the expressions

$$\sigma_{ij}(r,\theta) = K\tilde{\sigma}_{ij}(\theta)r^p, \quad D(r,\theta) = 1 - h(\theta)(r/r_0)^l \tag{29}$$

approximate expressions for p and l were derived as follows:

$$p = \frac{n-(m+1)}{n+m+1}, \qquad l = \frac{n-(m-1)}{n+m+1} \tag{30}$$

where n and m are exponents of the power creep law and the power creep damage law. For a uniform damage distribution $l = 0$, this relation leads to $p = -1/(n+1)$ and is reduced to the HRR field.

3. The above relations imply that the conditions $m-1 \leq n < m+1$ and $m+1 \leq n$ give the relations $p < 0$ and $p \geq 0$; i.e., singular and non-singular stress fields, respectively. This results furnishes very important information not only for evaluation and understanding of creep crack behavior, but also for discussion of stability and convergency in its numerical analyses.
4. While the asymptotic stress fields in the states of plane strain and plane stress have the identical stress singularity in undamaged materials, the state of plane strain has more significant stress singularity than that of plane stress in the case of preceding damage.

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